SM223 – Calculus III with Optimization
Assoc. Prof. Nelson Uhan

Fall 2017

Lesson 35. Triple Integrals

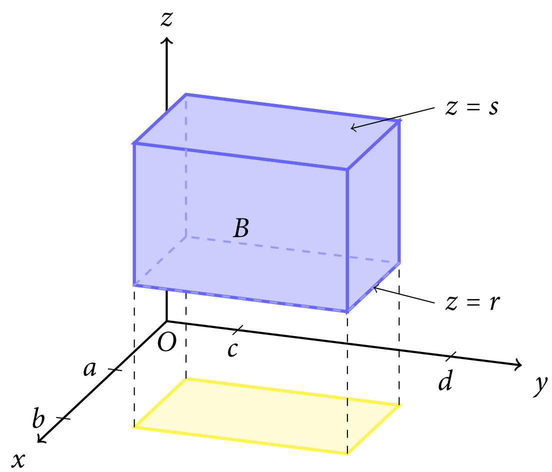
1 Overview

- Integrals of 3-variable functions over 3D regions of integration

2 Triple integrals over rectangular boxes

- **Fubini's theorem for triple integrals.** Let $B = \{(x, y, z) \mid a \leq x \leq b,\ c \leq y \leq d,\ r \leq z \leq s\}$. Then

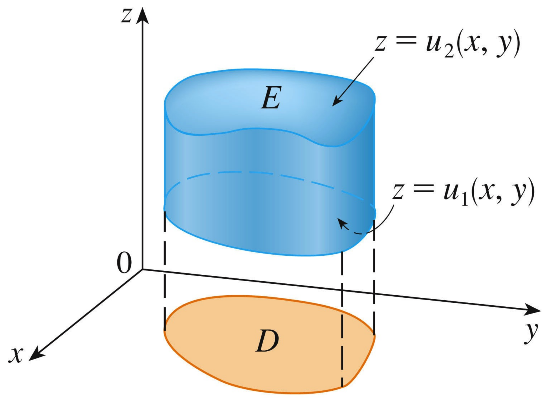
  - (f continuous on B)
  - Integrate from the inside out
  - When all limits of integration are constant, we can integrate in any order



Example 1. Evaluate the triple integral $\iiint_B x \, dV$, where B is the rectangular box given by $B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$.

3 Triple integrals over general bounded 3D regions

- **Type A 3D region:** between two continuous functions of x and y



  - E is the 3D region
  - D is the projection (shadow) of E onto the xy-plane
  - If E is a type A region, then

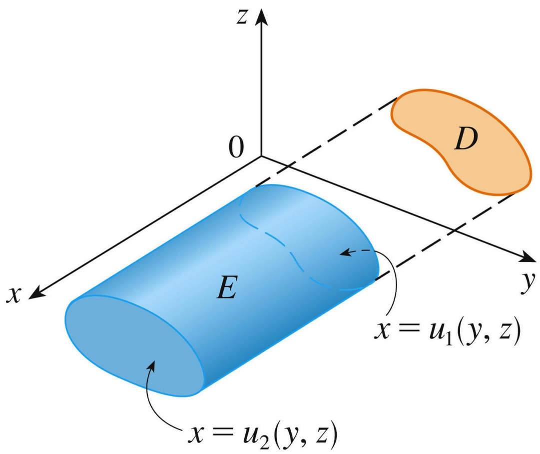
  - $(f, u_1, u_2$ continuous)
  - Integration from the inside out
  - Double integral over D can be done using previous techniques (e.g. Type I or II region)

Example 2. Express $\iiint_E x\, dV$ as an iterated integral, where E lies below the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = x^2$ and $y = x$.

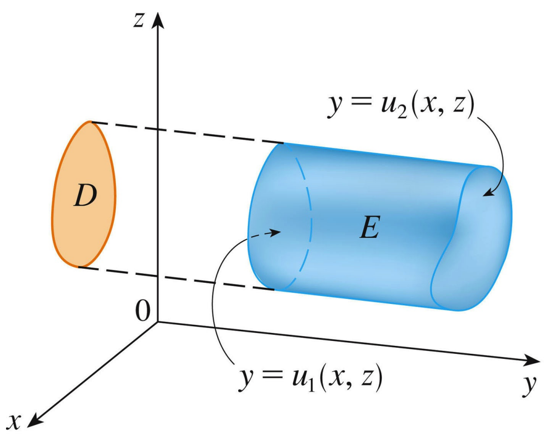
Example 3. Express $\iiint_E \sin(x + yz)\,dV$ as an iterated integral, where E lies below the surface $z = 1 + x^2 + 4y^2$ and above the region in the xy-plane bounded by the curves $x = 2y$, $x = 0$, and $y = 1$.

Example 4. Express $\iiint_E y\sqrt{z}\,dV$ as an iterated integral, where E is the solid tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$.

- **Type B 3D region:** between two continuous functions of y and z



- **Type C 3D region:** between two continuous functions of x and z



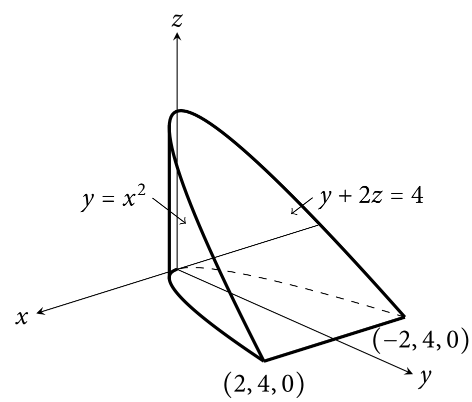
Example 5. Express $\iiint_E y\sqrt{z}\,dV$ as an iterated integral, where E is the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$. Consider E as a type B region.

Example 6. Express $\iiint_E y\sqrt{z}\,dV$ as an iterated integral, where E is the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$. Consider E as a type C region.

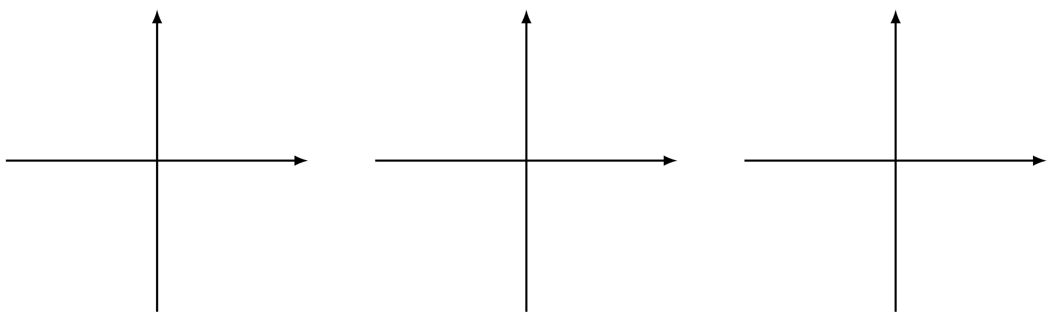
Example 7. Express $\iiint_E \sqrt{x^2 + z^2}\,dV$ as an iterated integral, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Example 8. The figure below shows the region of integration for the integral

$$\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{2-y/2} f(x,y,z)\,dz\,dy\,dx$$



a. Draw the projection of the region of integration onto the xy-plane, the yz-plane, and the xz-plane.



b. Rewrite the integral above as an equivalent iterated integral in the five other orders.